THERMAL GENERATION OF ELASTIC VIBRATIONS TAKING ACCOUNT OF THE FINITE HEAT-PROPAGATION VELOCITY

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UDC 536.24.02

By analyzing the solution of a problem of coupled dynamic thermoelasticity with the finite heat-propagation velocity taken into account, the possibility of its experimental determination by using acoustic methods is shown.

As is known, the acoustic characteristics of a substance are associated with its physicochemical properties, including the thermophysical properties. Hence, by studying acoustic vibrations originating in an elastic medium because of thermoelastic stresses, valuable information about the temperature and thermophysical properties of a substance, and, particularly, about the thermal relaxation due to the finite value of the heat-propagation velocity [1], can be obtained during periodic heating of the specimen. If the solution of the heat-conduction problem is analyzed for the case of periodic heating of a surface with the finite heat-propagation velocity taken into account, then it turns out that the amplitude, the attenuation coefficient, and the phase of the temperature waves depend on the magnitude of the heat velocity, where this dependence becomes clearer with the growth in the frequency of the heating source [2]. It is natural to expect that the amplitude, phase, and attenuation coefficient of the elastic vibrations which hence originate will also depend on the velocity of heat propagation.

To clarify the nature of this dependence it is necessary to solve a coupled thermoelasticity problem taking account of the finite heat-propagation velocity. Let us examine this problem for the one-dimensional case of a semiinfinite space. In contrast to a number of papers in which the pulse effect on a body is investigated, here we will study the case of a periodic, high-frequency heat flux, produced by a modulated continuous laser beam.

The problem is described mathematically by a system of equations [3] consisting of the generalized Fourier equation, the energy conservation equation, the equation of motion in an acoustic approximation, Hooke's law:

$$\begin{gathered}
\tau_r \frac{\partial q}{\partial t} + q = -\lambda \frac{\partial T}{\partial x}, \\
\rho c_v \frac{\partial T}{\partial t} + \rho \frac{c_p - c_v}{\alpha} \frac{\partial^2 u}{\partial t \partial x} = -\frac{\partial q}{\partial x}, \\
\frac{\partial^2 U}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \\
\frac{\partial U}{\partial x} = \frac{1}{1-\mu}\left[\frac{1-2\mu}{2G}\sigma + (1+\mu)\alpha T\right],
\end{gathered} \tag{1}$$

and the corresponding boundary conditions.

Let us write the system (1) in dimensionless form, as is done in [3] (the dimensionless variables are denoted by the same letters):

$$\frac{1}{b^2}\frac{\partial q}{\partial t} + q = -\frac{\partial T}{\partial x},$$

Belorussian Institute of Mechanization and Electrification of Agriculture. Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 3, pp. 538-543, September, 1975. Original article submitted July 7, 1974.

$$\frac{\partial T}{\partial t}+\delta\frac{\partial^2 U}{\partial x\partial t}=-\frac{\partial q}{\partial x}, \tag{2}$$

$$\frac{\partial^2 U}{\partial t^2}=\frac{\partial\sigma}{\partial x},\qquad \sigma=\frac{\partial U}{\partial x}-T.$$

We solve the problem under the following boundary conditions:

$$\lambda\frac{\partial T(0,\ t)}{\partial x}=q_0(\omega\tau_p\sin\omega t-\cos\omega t),\qquad U(0,\ t)=0, \tag{3}$$

where

$$\delta=\frac{(\gamma-1)(1+\mu)}{1-\mu},\qquad \gamma=\frac{c_r}{c_v},\qquad b=\frac{1}{c}\left(\frac{\lambda}{\rho c_v\tau_r}\right)^{1/2},$$

and c is the velocity of elastic volume wave propagation.

There are no displacements in the half-space at the initial instant and the temperature is constant; for simplicity, let us take $T(x, 0) = 0$. Since there are no internal sources and the velocity of all the perturbations is finite, all the partial derivatives with respect to the time are also zero everywhere at the time $t = 0$.

Let us apply the Laplace transformation to (2). Let us reduce the system which describes the problem to an equation for one function U^* (the Laplace transform of the displacement). Let us also write the boundary conditions in displacements by using the equations of the system (2). After simple manipulations, the mathematical formulation of the problem for U^* is written as

$$\frac{d^4U^*}{dx^4}-\left[p^2\left(1+\frac{1+\delta}{b^2}\right)+p(1+\delta)\right]\frac{d^2U^*}{dx^2}+U^*\left(\frac{p^4}{b^2}+p\right)=0, \tag{4}$$

$$U^*(0,\ p)=0,\qquad \frac{d^2U^*(0,\ p)}{dx^2}=\frac{q_0}{p^2+\omega^2}(\tau_r\omega^2-p). \tag{5}$$

As has been mentioned, the problem is examined in the half-space $x > 0$; hence the general solution is

$$U^*(x,\ p)=C_1\exp(-k_1x)+C_2\exp(-k_2x). \tag{6}$$

Here k_1 and k_2 are the arithmetical roots of the characteristic equation corresponding to the solution which attenuates at infinity,

$$k_{1,2}=\frac{1}{2}\left\{p^2\left(1+\frac{1+\delta}{b^2}\right)+p\left[1+\delta\pm\left(\left[p+\left(1+\frac{1+\delta}{b^2}\right)+1+\delta\right]^2-\frac{4p^2}{b^2}-4p\right)^{1/2}\right]\right\}.$$

Using the boundary conditions (5), the solution for the transform is written as

$$U^*(x,\ p)=\frac{1}{k_1^2+k_2^2}\cdot\frac{q_0}{p^2+\omega^2}(p-\tau_r\omega^2)(e^{-k_2x}-e^{-k_1x}). \tag{7}$$

Here $U^*(x, p)$ is an analytic function in the domain $\mathrm{Re}\,p > a_0$; it satisfies all the conditions of the inversion theorem and admits of extraction of a single-valued branch. Applying the inverse Laplace transform, we obtain

$$U(x,\ t)=\frac{1}{2\pi i}\int_{a_0-i\infty}^{a_0+i\infty}U^*(x_1p)\exp(pt)\,dp. \tag{8}$$

The solution (8) includes both the transient and steady parts of the process.

Since the length and time scales in the problem under consideration are small, it is of practical interest to extract the steady part of the solution from (8). The function (7) has four branch points (p_1, p_2, p_3, p_4) and two singular points of the simple pole type (p_5, p_6):

$$p_{1,2}=\frac{\left(1+\frac{1+\delta}{b^2}\right)(1+\delta)-2\pm 2i\sqrt{2}}{\left(1+\frac{1+\delta}{b^2}\right)^2-\frac{2}{b^2}},\qquad p_3=0,\quad p_4=-b^2,$$

$$p_{5,6}=\pm i\omega.$$

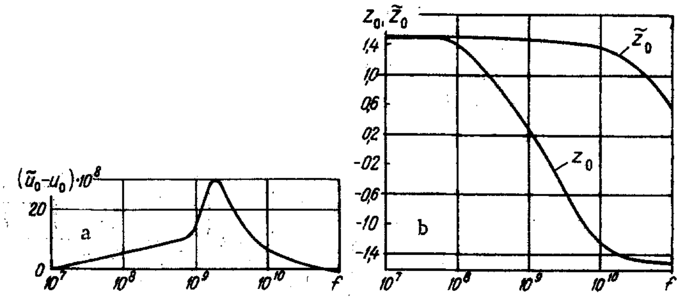

Fig. 1. Dependence of the difference in dimensionless amplitudes of acoustic vibrations (a) and additional phase shifts (b) on the frequency of the heat flux effect.

The process to obtain such a steady solution is described in [2], and hence we do not consider it further here, but write at once the steady solution for the displacement:

$$U(x,\ t) = U_0\left[\exp\left(-\frac{y_2}{2}x\right)\cos\left(\omega t - \frac{z_2}{2}x - z_0\right) - \exp\left(-\frac{y_1}{2}x\right)\cos\left(\omega t - \frac{z_1}{2}x - z_0\right)\right], \tag{9}$$

where

$$y_1 = \sqrt{d_1 + \sqrt{d_1^2 + f_1^2}};\quad z_1 = \sqrt{-d_1 + \sqrt{d_1^2 + f_1^2}};$$

$$y_2 = \sqrt{-g_1 + \sqrt{g_1^2 + h_1^2}};\quad z_2 = \sqrt{g_1 + \sqrt{g_1^2 + h_1^2}};$$

$$y_3 = \frac{1}{\sqrt{2}}\sqrt{m_1 + \sqrt{m_1^2 + r_1^2}};\quad z_3 = \frac{1}{\sqrt{2}}\sqrt{-m_1 + \sqrt{m_1^2 + r_1^2}};$$

$$m_1 = \omega^4\left[\left(1 + \frac{1+\delta}{b^2}\right)^2 - \frac{4}{b^2}\right] - \omega^2(1+\delta)^2;$$

$$r_1 = \omega^3\left[4 - 2\left(1 + \frac{1+\delta}{b^2}\right)(1+\delta)\right];$$

$$d_1 = y_3 - \omega^2\left(1 + \frac{1+\delta}{b^2}\right);\quad f_1 = z_3 + \omega(1+\delta);$$

$$g_1 = y_3 + \omega^2\left(1 + \frac{1+\delta}{b^2}\right);\quad h_1 = -z_3 + \omega(1+\delta);$$

$$z_0 = \operatorname{arctg}\frac{z_3 - \tau_r\omega y_3}{y_3 + \tau_r\omega z_3};\quad U_0 = q_0\sqrt{\frac{1+\omega^2\tau_p^2}{y_3^2 - z_3^2}}.$$

The solution (9) is two waves being propagated in the one direction x > 0 at different velocities and different attenuations. They are called fast and slow waves in the consideration of coupled thermoelasticity problems in contrast to the sound and heat waves in solutions of uncoupled problems.

The solution of an analogous problem without the finite velocity of heat propagation taken into account can be obtained from (9) by passing to the limit as the propagation velocity tends to infinity:

$$\tilde{U}(x,\ t) = \tilde{U}_0\left[\exp\left(-\frac{\tilde{y}_2}{2}x\right)\cos\left(\omega t - \frac{\tilde{z}_2}{2}x - \tilde{z}_0\right) - \exp\left(-\frac{\tilde{y}_1}{2}x\right)\cos\left(\omega t - \frac{\tilde{z}_1}{2}x - \tilde{z}_0\right)\right], \tag{10}$$

where

$$\tilde{y}_1 = \sqrt{d_2 + \sqrt{d_2^2 + f_2^2}};\quad \tilde{z}_1 = \sqrt{-d_2 + \sqrt{d_2^2 + f_2^2}};$$

$$\tilde{y}_2 = \sqrt{-g_2 + \sqrt{g_2^2 + h_2^2}};\quad \tilde{z}_2 = \sqrt{g_2 + \sqrt{g_2^2 + h_2^2}};$$

$$\tilde{y}_3 = \frac{1}{\sqrt{2}}\sqrt{m_2 + \sqrt{m_2^2 + r_2^2}};\quad \tilde{z}_3 = \frac{1}{\sqrt{2}}\sqrt{-m_2 + \sqrt{m_2^2 + r_2^2}};$$

$$m_2 = \omega^4 - \omega^2(1+\delta)^2;\quad r_2 = 2\omega^3[2 - (1+\delta)];$$

$$d_2 = \tilde{y}_3 - \omega^2;\quad f_2 = \tilde{z}_3 + \omega(1+\delta);\quad g_2 = \tilde{y}_3 + \omega^2;\quad h_2 = -\tilde{z}_3 + \omega(1+\delta);$$

$$z_0 = \operatorname{arctg}\frac{\tilde{z}_3}{\tilde{y}_3};\quad U_0 = \frac{q_0}{\sqrt{\tilde{y}_3^2 + \tilde{z}_3^2}}.$$

Since there is the possibility of heat flux in a broad frequency range acting on a semiinfinite body, it is interesting to examine how the acoustic wave parameters vary as the frequency of the heat flux effect changes. Let us consider the limit relationships when the frequency $\omega \to \infty$. However, it is here necessary to stipulate that the equations of a continuous medium are valid to frequencies on the order of 10^{10}-10^{11} Hz [4]. Hence, the limit relationships can be considered as an index of the direction of change of the appropriate acoustic parameters.

The attenuation coefficient y_1 tends to ∞ with the increase in frequency, while the attenuation coefficient y_2 tends to a finite value as $\omega \to \infty$:

$$y_2 = \frac{1+\delta-\sqrt{\left[2-\left(1+\frac{1+\delta}{b^2}\right)(1+\delta)\right]^2 \Big/ 2\left[\left(1+\frac{1+\delta}{b^2}\right)^2-\frac{4}{b^2}\right]}}{\sqrt{2\left[1+\frac{1+\delta}{b^2}+\sqrt{\left(1+\frac{1+\delta}{b^2}\right)^2-\frac{4}{b^2}}\right]}}. \tag{11}$$

Computations showed that one of the waves, namely, the one with the attenuation coefficient y_1, attenuates practically completely for a carbon steel specimen on the order of 0.1 mm long under a 10 MHz frequency of the heat flux effect.

It is also interesting that the phase velocity ν_1 grows with the increase in frequency, in contrast to the phase velocity ν_2 which tends to the finite value

$$\nu_2 = \frac{2}{\sqrt{2\left[\left(1+\frac{1+\delta}{b^2}\right)+\sqrt{\left(1+\frac{1+\delta}{b^2}\right)^2-\frac{4}{b^2}}\right]}},$$

i.e., this wave has no dispersion at high frequencies.

For the solution without taking account of τ_r, the attenuation coefficients and the phase velocities vary in the same manner with the increase in frequency. Such a nature of the change in phase velocities permits us conditionally to designate the wave with phase velocity ν_1 thermal, and the wave with phase velocity ν_2 acoustic.

The main purpose of this paper was to clarify the distinction between the solutions for the displacement with and without the finite velocity of heat propagation taken into account.

Comparing the solutions (9) and (10) showed that the vibration amplitudes of the solutions (9) and (10) tend to zero as $\omega \to \infty$. The ratio between the vibration amplitude of the solution with the heat-propagation velocity taken into account and the vibration amplitude without the heat-propagation velocity taken into account is not one. This indicates the distinct nature of the frequency dependence of the vibration amplitude.

The additional phase shift z_0 of the solution (9) tends to $(-\pi/2)$ as the frequency increases, while the additional phase shift $\tilde{z}_0$ of (10) diminishes as the frequency of the thermal effect increases and tends to zero in the limit.

The distinct nature of the attenuation should be noted. Thus, the attenuation coefficient y_1 of the heat wave (9) tends to infinity as ω with the increase in frequency, but the attenuation coefficient of the heat wave (10) tends to infinity as $\sqrt{\omega}$ with the change in frequency. The limits of the acoustic wave attenuation coefficients tend to the finite values y_1 (11) and $\tilde{y}_2$, where

$$\tilde{y}_2 = \frac{(\sqrt{2}-1)+(\sqrt{2}+1)\delta}{2\sqrt{2}}.$$

Therefore, the acoustic-wave parameters obtained taking account of the finite heat rate differ, for a sufficiently high frequency of the thermal effect, from the corresponding parameters when the heat-propagation velocity is assumed infinite.

The solutions obtained were computed on a "Minsk-22" electronic digital computer. In the investigations we used carbon steel at room temperature, whose relaxation time was taken as $\tau_r = 10^{-10}$ sec. The amplitude of the heat flux was $q_0 = 1$ W/cm^2. The results of the computation are presented as graphs.

Shown in Fig. 1a is the difference between the dimensionless vibration amplitudes. The maximum difference is achieved at $\omega\tau_r = 1$ and is on the order of 10^{-5} Å. These are practically indistinguishable amplitudes. The vibration amplitude itself is 10^{-3} Å at an $f = 10^8$ Hz frequency of the effect.

As is seen from Fig. 1b, the nature of the change in the additional phase shifts is substantially distinct. This affords the possibility of determining the relaxation time and the magnitude of the heat velocity, respectively, by means of experimentally measured values of the amplitude, the attenuation coefficient, and the phase of the periodic component of the heat flux.

NOTATION

T	is the temperature;
q	is the heat flux;
U	is the displacement in the x direction perpendicular to the surface of the half-space ($x > 0$);
σ	is the normal stress in a plane parallel to the surface;
λ	is the coefficient of thermal conductivity;
ρ	is the density;
c_p, c_v	are the specific heats at constant pressure and at constant volume;
α	is the coefficient of thermal expansion;
μ	is the Poisson's ratio;
G	is the shear modulus;
c	is the velocity of elastic volume wave propagation;
k_1, k_2	are the arithmetic roots of the characteristic equation corresponding to the solution attenuating at infinity;
ω	is the cyclic vibration frequency;
τ_r	is the relaxation time;
z_0	is the phase shift independent of the distance,
δ	is the coupling parameter;
U_0	is the vibration amplitude.

LITERATURE CITED

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